UDC 531.36: 534

QUALITATIVE STRUCTURES AND BIFURCATIONS GENERATED BY A NONLINEAR THIRD-ORDER PHASE SYNCHRONIZATION EQUATION

PMM Vol. 42, No. 5, 1978, pp 808-819
V. N. BELYKH and V. I. NEKORKIN
(Gor'kii)
(Received July 11, 1977)

A qualitative investigation is made of a system of nonlinear third-order differential equations, being a model of a phase synchronization system. The existence is established of bifurcation surfaces separating the parameter space into domains for whose points the system is globally asymptotically stable, contains cycles, has a complex structure (contains a denumerable set of saddle cycles), etc.

1. Introduction. Basic results. The task of analysing a typical phase synchronization system reduces to the investigation of an operator equation of the form [1].

$$p\varphi + K(p)[F(\varphi) - \gamma] = 0, \quad p \equiv d / dt \tag{1.1}$$

where φ is the phase, $F(\varphi)$ is a periodic nonlinearity, $K(p)$ is the transfer function of a low-frequency filter. The task of a qualitative investigation is the complete separation of the parameter space into domains corresponding to different qualitative trajectory patterns in phase space and has been solved in the case of a second-order Eq. (1.1) [2, 3] of the form

$$\varphi^{\cdot\cdot} + (\lambda + aF')\varphi^{\cdot} + F(\varphi) = \gamma \tag{1.2}$$

for the class of sinusoidal functions occuring in applications.

In the case of Eq. (1.1) of third order and higher this task of "complete separation" can become meaningless because in principle complex structures can exist in the phase space, while domains filled with an infinite set of bifurcations [4] can exist in the parameter space. Here, instead of a complete separation it is possible to state the qualitative investigation problem in the following way:

Problem 1. Determine a separation of the parameter space into domains in each of which the dynamic systems is either a Morse—Smale system (a structurally-stable system with a finite number of equilibrium states and periodic motions) or a system with an infinite set of periodic motions.

This problem relative to (1.1) is of interest in connection with separation the domain K of parameters, for whose points Eq. (1.1) is globally asymptotically stable, and the parameter domains adjoining it (see [5-7] and others for sufficient conditions for the global asymptotic stability of Eq. (1.1)). The question on the bifurcation of a saddle separatrix loop of a third-order Eq. (1.1) with a transfer function $K(p)$ (written in a form suitable for use subsequently)

$$K(p) = (ap+1)(bp^2+p+\lambda)^{-1} \tag{1.3}$$

was numerically analyzed in [8, 9] by the adjusting method. However, both Problem 1 as well as the question on the boundary of domain K were unresolved in [8, 9].

In the degenerate case of Eqs. (1.1) and (1.3) with $\lambda = 0$, corresponding to the synchronization system sought, when domain K is absent for $\gamma > 0$, a qualitative investigation of (1.1) and (1.3) was made in [10]. For an Eq. (1.1) of arbitrary order and, in particular, for the case of (1.3) with $b < a$ and certain other additional constraints, the existence of principal separation domains in the sense of Problem 1 was established in [11] by using matching systems of lower dimension.

Below we investigate qualitatively a third-order system (1.1), (1.3) of the form

$$\varphi^{\cdot} = y,\ y^{\cdot} = z,\ z^{\cdot} = b^{-1}[\gamma - F(\varphi) - \lambda y - aF'y - z] \tag{1.4}$$

in the domain $D = \{\gamma, a, b, \lambda\}$ of positive parameters. We assume that the function $F(\varphi) \in C^3$ and satisfies the conditions

$$\begin{gathered} F(\varphi) = F(\varphi+2\pi),\ -F(\varphi) = F(-\varphi),\ F'(\varphi) > 0,\ \varphi \in (-\varphi_0, \varphi_0) \\ F'(\varphi) < 0, \quad \varphi \in (\varphi_0, 2\pi - \varphi_0), \quad F'(\varphi_0) = 0, \quad F(\varphi_0) = 1 \\ F''(\varphi) < 0, \quad \varphi \in [\varphi_0, \pi), \quad F'''(\varphi) < 0, \quad \varphi \in (-\varphi_0, \varphi_0) \end{gathered} \tag{1.5}$$

Under these conditions system (1.4) with $\gamma < 1$ has two equilibrium states: O_1 ($\varphi = \varphi_1(\gamma)$, $y = z = 0$) and O_2 ($\varphi = \varphi_2(\gamma)$, $y = z = 0$), where φ_1 and φ_2 are roots of the equation $\gamma - F(\varphi) = 0$ on the half-open intervals $[0, \varphi_0)$ and $(\varphi_0, 2\pi]$, respectively. The system's phase space $G = S^1 \times R^2$ is cylindrical. We introduce the notation $m(\gamma) \equiv F'(\varphi_1)$ and $n(\gamma) \equiv -F'(\varphi_2)$. We shall designate nonwandering trajectories of oscillatory type as o-trajectories, of rotary type with positive rotation of phase φ as φ^1-trajectories, and of rotary type with negative (reverse) rotation of phase φ as φ^2-trajectories.

When $b = 0$ system (1.4) degenerates into a two-dimensional system corresponding to Eqs. (1.2), for which the separation of the parameter space $D_0 = \{\lambda \geqslant 0,\ \gamma \geqslant 0,\ -\infty < a < \infty\}$ has been obtained in [3] (for $F = \sin\varphi$ such a separation in the plane λ, $d = -a\lambda^{-1}$ was given by Bautin [2]). Since this separation is used below, in Fig. 1 we present its qualitative form in the plane (λ, γ), $a =$ const for *(1)* $a > 0$, *(2)* $a = 0$, *(3)* $a < 0$. The structures of the separation of cylinder $S^1 \times R$ by the trajectories of Eq. (1.2) is shown in Fig. 2 (the structures labelled by the letters $K, L_3, \ldots$, correspond to the parameter domains labelled by the same letters in Fig. 1). Structure K corresponds to the global asymptotic stability of Eq. (1.2). The bifurcation surfaces Π_0^1, Π_0^2 and Π_0° correspond to the bifurcations of the φ^1-, φ^2-and o-loops of the separatrix, respectively.

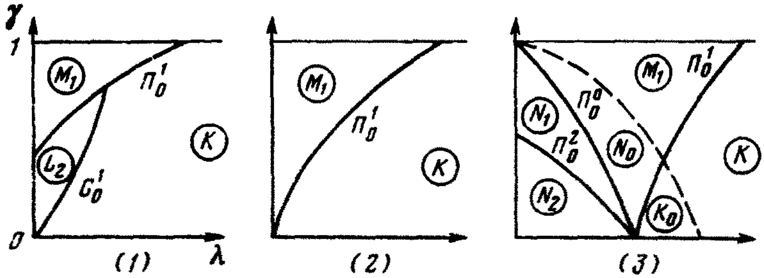

Fig. 1

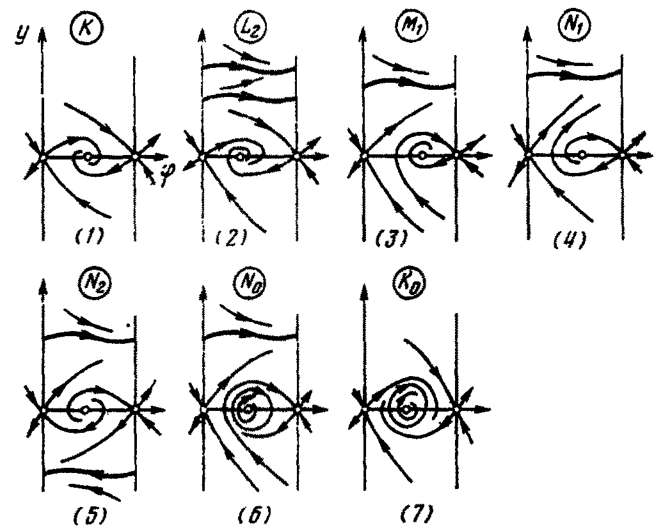

Fig. 2

When $a > 0$ the saddle index of the saddle changes sign and divides Π_0^1 into two parts, for one of which a stable φ^1-cycle goes into φ^1-loop, while for the other an unstable φ^1-cycle is generated from the φ^1-loop, merging then with a stable φ^1-cycle on passing through the surface C_0^1. For $\lambda = -am(\gamma)$, $a < 0$ (the dashed line in Fig. 1(3)), a change of stability of O_1 occurs, as a result of which a stable (the Liapunov index is negative) o-cycle is generated, going into an o-loop.

As a result of investigating system (1.4) it has been established that the bifurcation surfaces Π_0^j (here and below, $j = 0, 1, 2$) are preserved under an increase of parameter b from zero. More precisely, there holds

T h e o r e m 1. For each function $F(\varphi)$ satisfying (1.5), in D there exist surfaces $\Pi_b^j = \{\gamma, a, b, \lambda \mid \rho_j(a, b, \lambda, \gamma) = 0\}$ corresponding to the o-, φ^1- and φ^2-loops of the separatrix of the saddle of system (1.4). The disposition of

Π_b^1 on the plane (λ, γ) $(b, a = \text{const})$ for $b < a$ qualitatively coincides with the disposition of the bifurcation curve Π_0^1 of system (1.2) with $a > 0$, for $b = a$ it coincides with the disposition of curve Π_0^1 of system (1.2) with $a = 0$, while for $b > a$ the disposition of Π_b^j coincides with the disposition of Π_0^j of system (1.2) with $a < 0$.

Theorem 1 is proved in Sect. 3.

The different nature of the behavior of the trajectories of system (1.4) under the collapse of the separatrix loops in a neighborhood of surface Π_b^j is determined by Shil'nikov's theorems [12, 13]. We separate the parameter space D into three domains

$$\sigma^s = \{\gamma, a, b, \lambda \mid 0 \leqslant \gamma < 1,\ b \geqslant a,\ \lambda > n(\gamma)(b + a) - 2b^{-1}\}$$

$$\sigma^u = \{\gamma, a, b, \lambda \mid n(\gamma) > b^{-2},\ b \geqslant a, \lambda < f(a, b, n), a^2 > (n^\circ)^{-1}\},$$

$$n^\circ \equiv \sup_{\gamma \in [0, 1)} n(\gamma)$$

$$\sigma^c = \{\gamma, a, b, \lambda \mid n(\gamma) > b^{-2},\ b \geqslant a,\ f(a, b, n) < \lambda < n(\gamma)(b + a) - 2b^{-1}\}$$

where $f(a, b, n)$ is a positive root of the equation $4b(\lambda - an)^3 - (\lambda - an)^2 + 18bn(\lambda - an) + 27n^2b^2 - 4n = 0$ relative to λ. If these domains divide the bifurcation surface of a separatrix loop into three pieces, then, according to [12, 13], the bifurcation properties of each of them having the codimension 1 are different. Figure 3 illustrates these properties by example of system (1.4) with $b = a$ and $a^2 > (n^0)^{-1}$, when the bifurcation surface Π_b^1 intersects all three domains σ^s, σ^u and σ^c. In particular, the condition $\Pi_b^j \cap \sigma^c \neq 0$ yields the existence of domain $d_c^j = \{\gamma, a, b, \lambda \,\|\, \rho_j(a, b, \lambda, \gamma)| < \varepsilon\} \cap \sigma^c$ containing an infinite set of bifurcations. For parameter values from domain d_c^j system (1.4) has a complex trajectory structure containing, respectively, a denumerable set of saddle o-, φ^1- and φ^2-cycles.

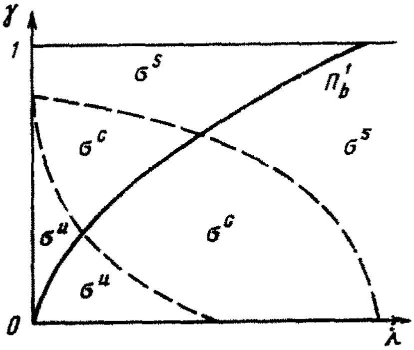

Fig. 3

Using the properties of equilibrium states and of separatrix loops, Lemma 2.1—2.3, and Theorem 2 on the limit set (for brevity it is called a stable φ^1 (φ^2)-cycle in [11]), with the aid of a continuous variation of parameters from some structures to others we establish the existence of the following bifurcation sets distinct from Π_b^j.

1) When $b > a$ in D exists a surface R corresponding to the change of stability of equilibrium state O_1 and to the generation of a stable o-cycle. When $b, a = \text{const}$ the disposition of R on the plane (λ, γ) qualitatively coincides

with the disposition of curve $\lambda = -am(\gamma)$ $(a < 0)$ of system (1.2) (the dashed line in Fig. 1(3)).

2) In domain σ^u exist bifurcation sets C_b^j on which befurcations of the o-, φ^1 and φ^2-cycles take place. The bifurcation sets C_b^j are bifurcation surfaces of multiple cycles if the number of cycles in the toroidal domains is taken mod 2.

3) Under a variation of parameters from domains d_c^j to domains Δ_i $(i = 1, 2, \ldots, 7)$ in which the corresponding cycles are absent, the complex structures vanish. Consequently, bifurcation sets exist, being the boundaries of domains $D_c^j \supset d_c^j$ corresponding to the complex structures of system (1.4). The nature of the boundaries of domains D_c^j is not clear, just as the relative disposition of these domains and of sets C_b^j is not clear.

Let us separate space D, assuming for simplicity that C_b^j is single-valued. If the surfaces Π_b^j are wholly located in domains σ^s and σ^u (for instance, when $a^2 < (n^\circ)^{-1}$, $b \leqslant -2^{-1}a + [4^{-1}a^2 + 2(n^\circ)^{-1}]^{1/2}$; $a^2 = (n^\circ)^{-1}$, $b = a$), then the surfaces C_b^j are additional bifurcations. In this case the qualitative form of the separations when $b, a = \text{const}$ is shown in Fig. 1 for *(1)* $b < a$, *(2)* $b = a$, *(3)* $b > a$. It is the same as in the two-dimensional case when $b = 0$ (system (1.2)). The surface C_b^1 adjoins Π_b^1 on the set $\Pi_b^1 \cap \Gamma$, where Γ is the common boundary of domains σ^u and σ^s (Fig. 1(1)). In addition, C_b^1 can also adjoin Π_b^1 at points for which in G the separatrix W_1^u approaches saddle O_2 along a nonfundamental direction. To denote the separation domains of space D and the separation structure of space G corresponding to these domains we retain the notation of the corresponding domains and structures of the two-dimensional system: K, L_2, . . (see the description of structures K, L_2, . . . in [11]). When surface Π_b^j is located in all three domains σ^u, σ^s and σ^c (for instance when $a^2 > (n^\circ)^{-1}$, $b = a$), the separation of D is shown in Fig. 4 for *(1)* $b < a$, *(2)* $b = a$, *(3)* $b > a$. The domains D_c^j are shown hatched, while the bifurcation surfaces C_b^j are by convention depicted adjoining D_c^j. The qualitative patterns for domains *1*, *2*, . . ., *7* in Fig. 4 are easily re-established by passing into them from the domains K, M_1, L_2, . . . through the appropriate bifurcations. For example, four cycles are generated by passing from K_0 into domain 2 through point c : two stable φ^1- and φ^2-cycles and two saddle φ^1- and φ^2-cycles. Domains 1 –7 vanish as parameter b increases.

The parameter domain K corresponding to the capture domain of the phase synchronization system is delineated by bifurcation surfaces: of the separatrix loop (Π_b^1), of the multiple φ^1-cylces (C_b^1), of the change of stability of equilibrium state O_1 (R), and by the bifurcations of the φ^1-trajectories, leading to the origin of the complex structure. Thus, we have succeeded in obtaining an overall separation of the parameter space of system (1.4) as a whole by using the proof of the existence of separatrix loops, Lemmas 2.1 –2.3, Theorem 2 and assertions from [12, 13], although with certain restrictions connected with the principal difficuties inherent in multidimensional systems.

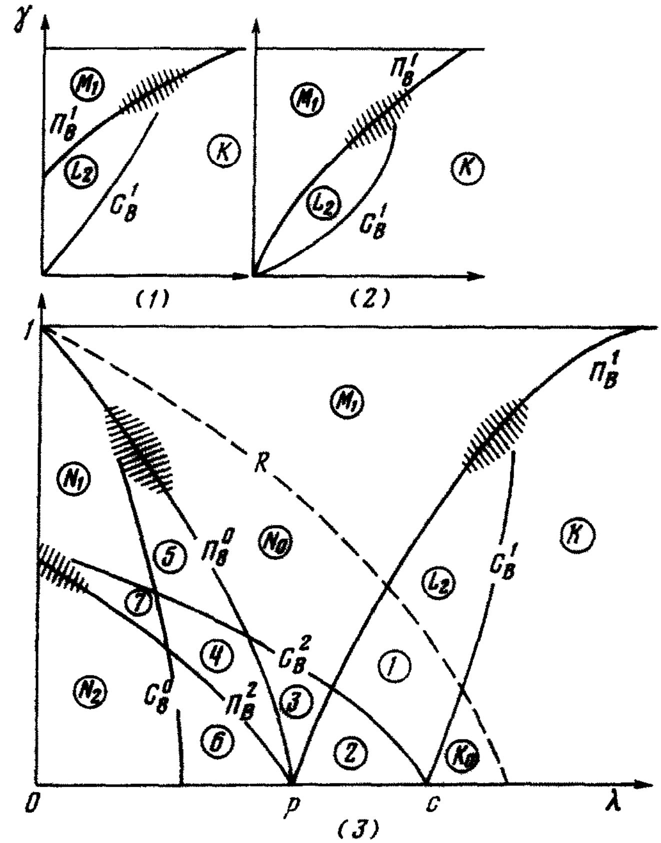

Fig. 4

2. Investigation of the system. System (1.4) is examined below in the parameter domain $D = \{\gamma, a, b, \lambda \mid 0 \leqslant \gamma < 1,\ b \geqslant a,\ \lambda > 0\}$. By a change of variables z and y and of time

$$z = v - b^{-1}\delta y - ab^{-1}[F(\varphi) - \gamma], \quad y = a^{1/2}b^{-1/2}y_H, \quad t = b^{1/2}a^{-1/2}\tau$$

retaining the previous notation, system (1.4) is transformed to the form:

$$\varphi^{\cdot} = y, \quad y^{\cdot} = \gamma - F(\varphi) - (ab)^{-1/2}\delta y + ba^{-1}v \qquad (2.1)$$

$$v^{\cdot} = -(ab)^{-1/2}(1 - \delta)v - b^{-2}(\delta^2 - \delta + \lambda b)y + b^{-1}(ab)^{1/2}(a - b - a\delta)[F(\varphi) - \gamma]$$

$$\delta = \begin{cases} 2^{-1} - (4^{-1} - \lambda b)^{1/2}, & \lambda \leqslant (4b)^{-1} \\ \lambda a, & \lambda > (4b)^{-1} \end{cases}$$

Let us analyze system (2.1) equivalent to system (1.4). For $\lambda > (4b)^{-1}$ we consider the matching system

$$y^{\cdot} = -a^{1/2}b^{-1/2}\lambda y + ba^{-1}v + \gamma + \operatorname{sgn} v \qquad (2.2)$$
$$v^{\cdot} = -(ab)^{-1/2}(1 - \lambda a)v + \lambda b^{-2}(a - b - \lambda a^2)y -$$
$$b^{-1}(ab)^{-1/2}(a - b - \lambda a^2)(\gamma + \operatorname{sgn} v)$$

System (2.2) is a piecewise-linear system, continuously spliced when $v = 0$, specified on the phase surface (y, v). The trajectories of system (2.2) generate a mapping of the segment $\eta = \{y, v \mid y \in (-\lambda^{-1}a^{-1/2}b^{1/2}(1 - \gamma), -\infty), v = 0\}$ onto itself. It can be shown that this mapping has a unique stable fixed point corresponding to the asymptotically stable cycle C° of system (2.2), whose equations in parametric form are

$$v = \operatorname{sgn} v r^{-1}b^{-2}A\lambda(\lambda a^2 + b - a)\exp[-2^{-1}(ab)^{-1/2}\theta]\sin r\theta \qquad (2.3)$$
$$y = \operatorname{sgn} vA[2^{-1}(ab)^{-1/2}r^{-1}(2\lambda a - 1)\sin r\theta -$$
$$\cos r\theta]\exp[-2^{-1}(ab)^{-1/2}\theta] + \lambda^{-1}a^{-1/2}b^{1/2}(1 + \gamma \operatorname{sgn} v)$$
$$A \equiv 2\lambda^{-1}b^{1/2}a^{-1/2}\{1 - \exp[-\pi(4\lambda b - 1)^{-1/2}]\}^{-1}$$
$$r \equiv 2^{-1}(ab)^{-1/2}(4\lambda b - 1)^{1/2}, \quad 0 \leqslant \theta \leqslant \pi$$

The trajectories of system (2.2) in space G form cylindrical surfaces whose directrices are formed by the trajectories of system (2.2), while the generators are parallel to the φ-axis. By G_v we denote the sub-domain of G, bounded for $\lambda > (4b)^{-1}$ by the surface $T^{\circ} = C^{\circ} \times S^1 = \{\varphi, y, v \mid \varphi \in S^1, y, v \in C^{\circ}\}$ and for $\lambda \leqslant (4b)^{-1}$ by the surface $T^{\circ} = \{\varphi, y, v \mid \varphi \in S^1, y = y_i^{\circ}, v = v_i^{\circ}\}$, where

$$y_i^{\circ} = (-1)^i a^{-1}\delta^{-1}(1 - \delta)^{-1}b(ab)^{1/2}(1 + (-1)^i\gamma)$$
$$v_i^{\circ} = (-1)^i b^{-1}(1 - \delta)^{-1}(a\delta + b - a)(1 + (-1)^i\gamma), \quad i = 1, 2$$

L e m m a 2.1. The set Ω of nonwandering trajectories of system (2.1) is wholly contained in domain G_v. The vector field of system (2.1) on T° is directed to the interior of G_v.

P r o o f. For $\lambda \leqslant (4b)^{-1}$ we consider the direction function $w = 2^{-1}v^2$. Outside the domain $G_1 = G \cap \{v_1^0 \leqslant v \leqslant v_2^0\}$ the derivative $w^{\cdot}$, taken relative to system (2.1), is negative and, consequently, the trajectories outside domain G_1 as $t \to \infty$, intersecting the levels $v = \text{const}$, pass into G_1 and do not leave it. Then for any semi-trajectory $(\varphi(t), y(t), v(t)) \in G_1$ the fulfilment of the inequalities $(-1)^i y^{\cdot}(t) < 0$ for $(-1)^i y > (-1)^i y_i^0$ follows from the second equation in (2.1). From here and from the behavior of the trajectories of system (2.1) outside domain G_1 follows the lemma's assertion when $\lambda \leqslant (4b)^{-1}$. Consider the domain $\lambda > (4b)^{-1}$. We compare the vector fields of systems (2.2) and (2.1)

$$\left(\frac{dy}{dt}\right)_{(2.2)} - \left(\frac{dy}{dt}\right)_{(2.1)} = \operatorname{sgn} v + F(\varphi) \qquad (2.4)$$

$$\left(\frac{dv}{dt}\right)_{(2.2)} - \left(\frac{dv}{dt}\right)_{(2.1)} = b^{-1}(ab)^{-1/2}(\lambda a^2 + b - a)[\operatorname{sgn} v + F(\varphi)]$$

From (2.4) we get that the trajectories of system (2.1) outside G_v, intersecting without contact the cylindrical surfaces formed by the trajectories of system (2.2) as $t \to \infty$, pass into G_v and do not leave it.

Corollary. The inequalities

$$-\mu_1 < ba^{-1}v(t) < \mu_2, \quad \lambda \leqslant (4b)^{-1}$$
$$|ba^{-1}v(t)| < \mu_3, \quad \lambda > (4b)^{-1} \tag{2.5}$$

$$\mu_i = \frac{(b - a + a\delta)}{a(1-\delta)}(1 + (-1)^i\gamma), \quad i = 1, 2$$

$$\mu_3 = \frac{A\sqrt{\lambda}(\lambda a^2 + b - a)}{b\sqrt{a}} \exp\left[-(4\lambda b - 1)^{-1/2} \operatorname{arctg}(4\lambda b - 1)^{1/2}\right]$$

are valid for any semitrajectory $(\varphi(t), y(t), v(t)) \in G_v$ of system (2.1).

Equilibrium states. When $\lambda > m(\gamma)(b - a)$ the point O_1 is stable ($\varkappa_3^{(1)} < 0$, $\operatorname{Re}\varkappa_{1,2}^{(1)} < 0$), while when $\lambda < m(\gamma)(b - a)$ it is unstable ($\varkappa_3^{(1)} < 0$, $\operatorname{Re}\varkappa_{1,2}^{(1)} > 0$) ($\varkappa_n^{(1)}$ and $\varkappa_n^{(2)}$ are the roots of the characteristic equation of system (2.1) for O_1 and O_2, respectively, $n = 1, 2, 3$). The variation of the qualitative structure of the neighborhood of equilibrium state O_1 when passing into D through the surface $R = \{\gamma, a, b, \lambda \mid 0 \leqslant \gamma < 1, b > a, \lambda = m(\gamma)(b - a)\}$ is determined by the sign of the first Liapunov index [14] which in the case being examined has the form

$$L = C\,[a^{-2}m(1 + 4abm)F'''(\varphi_1) - ((ab)^{-1/2} + 8m(b - a) + 6bm)(F''(\varphi_1))^2], \quad C > 0 \tag{2.6}$$

By virtue of (1.5), $L < 0$ and, consequently, under the change in stability a single stable o-cycle is generated from O_1.

The equilibrium state O_2 is a saddle: when $\lambda > f(a, b, n)$ the point O_2 is a saddle-focus ($\varkappa_3^{(2)} > 0$, $-\operatorname{Re}\varkappa_{1,2}^{(2)} < 0$, $\operatorname{Im}\varkappa_{1,2}^{(2)} \neq 0$), while when $\lambda < f(a, b, n)$ it is a saddle ($\varkappa_3^{(2)} > 0$, $-\operatorname{Re}\varkappa_{1,2}^{(2)} < 0$, $\operatorname{Im}\varkappa_{1,2}^{(2)} = 0$). As is well known, two local manifolds pass through O_2 into its neighborhood: the two-dimensional surface W_l^s, consisting of O^+-curves, and the unstable W_l^u, being a one-dimensional O^--curve passing through O_2 and consisting of two separatrices $W_{l_2}^u$ and $W_{l_1}^u$ located on different sides of W_l^s. Analysis of system (2.1) linearized in a neighborhood of O_2 shows that the separatrix $W_{l_1}^u$ goes into the domain $G_v^1 = G_v \cap \{y > 0\}$ and $W_{l_2}^u$ goes into $G_v^2 = G_v \cap \{y < 0\}$. The extensions of local manifolds W_l^s and W_l^u of equilibrium state O_2 by the trajectories of system (2.1) are denoted W^s and $W^u = W_1^u \cup O_2 \cup W_2^u$, respectively.

The Liapunov function. We introduce the notation for subdomains of D

$$\Delta_1 = \{\gamma, a, b, \lambda \mid \gamma = 0,\ b = a, \lambda > 0\}$$
$$\Delta_2 = \{\gamma, a, b, \lambda \mid \gamma = 0,\ b > a,\ \lambda \geqslant b \sup_{\varphi \in [0, 2\pi)} F'(\varphi)\}$$

$$\Delta_3 = \{\gamma, a, b, \lambda \mid 0 < \gamma < 1,\ b = a,\ \lambda > 0\}$$
$$\Delta_4 = \{\gamma, a, b, \lambda \mid 0 < \gamma < 1,\ b > a,\ \lambda \geqslant b \sup_{\varphi \in [0, 2\pi)} F'(\varphi)\}$$

We consider a Liapunov function [5] and its derivative relative to system (2.1)

$$V = V^\circ + \beta \int_{\varphi_1}^{\varphi} [F(\xi) - \gamma]\, d\xi$$

$$V^\circ = \begin{cases} 2^{-1}\lambda a^2 y^2 + 2^{-1} a (av - \delta y)^2, & \beta = a^2\lambda,\ b = a \\ 2^{-1} b^{-2} [b^2 v + (ab)^{1/2}(1 - \delta) y]^2 + 2^{-1}\lambda a y^2 + \\ + (ab)^{1/2} [F(\varphi) - \gamma] y, & \beta = 1 + \lambda a,\ b > a \end{cases}$$

$$V^\cdot = \begin{cases} -(av - \delta y)^2 & b = a \\ -b^{-1}(ab)^{1/2}(\lambda - bF'(\varphi)) y^2 - (ab)^{1/2}[F(\varphi) - \gamma]^2, & b > a \end{cases}$$

We obtain the next statement with the aid of this function.

L e m m a 2.2. System (2.1)

1. has the structure K in the parameter domains Δ_1 and Δ_2;
2. does not have o-cycles in Δ_3 and Δ_4 , while the domain

$$\Omega^+ = \{\varphi, y, v \mid V(\varphi, y, v) < C_1,\ \varphi < \varphi_2\} \quad (C_1 \equiv V_1(\varphi_2, 0, 0))$$

belongs to the domain of attraction of the stable equilibrium state O_1;

3. does not have φ^2-cycles in Δ_3 and Δ_4,

M a t c h i n g s y s t e m. We consider the auxiliary system

$$\varphi^\cdot = y, \quad y^\cdot = \gamma_* - F(\varphi) - \lambda_* y, \quad v^\cdot = 0 \tag{2.7}$$

For each $v = \text{const}$ system (2.7) is system (1.2) with $a = 0$. The separation of the parameter plane (λ_*, γ_*) has been shown in Fig. 1(2). The bifurcation curve $\gamma_1(\lambda_*)$ $(\gamma_1(\lambda^\circ) = 1)$ corresponding to the φ^1-loop divides the plane (λ_*, γ_*) into two domains K and M_1. In space G_v the trajectories of system (2.7) form surfaces not changing with respect to v . By W^l we denote the surface formed by the cycle of system (2.7), existing in parameter domain M_1, while by W_1^α and W_1^ω (W_2^α and W_2^ω) we denote parts of the surfaces formed by the α - and ω-separatrices of the saddle of system (2.7), located in the domain G_v^1 (G_v^2). We examine the surfaces W_i^α and W_i^ω before intersection with cylinder $P_0 = \{\varphi, y, v \mid \varphi \in S^1,\ v \in R^1,\ y = 0\}$.

In D we introduce the subdomains

$$\Delta_5 = \left\{\gamma, a, b, \lambda \mid \gamma_1\left(\frac{\delta}{\sqrt{ab}}\right) + \mu_i < \gamma < 1, \frac{\delta}{\sqrt{ab}} < \lambda^\circ\right\}$$

$$\Delta_6 = \left\{\gamma, a, b, \lambda \mid -1 + \mu_i < \gamma < 1, \frac{\delta}{\sqrt{ab}} \geqslant \lambda^\circ; -\gamma_1\left(\frac{\delta}{\sqrt{ab}}\right) + \right.$$

$$\mu_i < \gamma < \gamma_1\left(\frac{\delta}{\sqrt{ab}}\right) + \mu_i, \frac{\delta}{\sqrt{ab}} < \lambda^\circ\Big\}$$

$$\Delta_7 = \Big\{\gamma, a, b, \lambda \,|\, 0 < \gamma < 1 - \mu_l, \frac{\delta}{\sqrt{ab}} \geqslant$$

$$\lambda^\circ 0 < \gamma < \gamma_1\left(\frac{\delta}{\sqrt{ab}}\right) - \mu_l, \frac{\delta}{\sqrt{ab}} < \lambda^\circ\Big\}$$

$i = 1, 2;\ l = 2, 3;$

L e m m a 2.3. In the parameter domains Δ_5 and Δ_6 (Δ_7) system (2.1) does not have φ^2 (φ^1)-cycles; as $t \to -\infty$ the manifold W^s, intersecting T°, goes off to infinity in domain G_v^2 (G_v^1).

P r o o f. Let us compare the vector fields of systems (2.1) and (2.7) when $\lambda_* = (ab)^{-1/2}\delta$ and $\gamma_* = \gamma - \mu_i$ ($\gamma_* = \gamma + \mu_l$). Using Lemma 2.1, we get that in the parameter domains Δ_5 and Δ_6 (Δ_7) the trajectories of system (2.1) intersect the surfaces W_i^ω and W_i^α on the side of increase (decrease) of coordinate y, without having contact with them. We establish the lemma's assertion by taking into account that surface W_2^ω (W_1^ω) goes off to infinity in domain G_v^2 (G_v^1) (Fig. 2(1), (3)), while by virtue of the equation $\varphi^\cdot = y$ in (2.1) the coordinate φ decreases (increases) along the trajectories when $y < 0$ ($y > 0$) .

3. B i f u r c a t i o n s a n d c y c l e s. Examining system (2.1) in the space $G = \{\varphi, y, v \,|\, \varphi_0 \leqslant \varphi \leqslant \varphi_0 + 2\pi,\ (y, v) \in R^2\}$, we prove an auxiliary lemma.

L e m m a 3. 1. In parameter domain D the separatrix W_2^u intersects the plane $P_1 = \{y, v \,|\, \varphi = \varphi_0, (y, v) \in R^2\}$ on the side $\varphi > \varphi_0$ at the point $M_2(y_2, v_2) \in G_v^2$ and the separatrix W_1^u intersects the plane $P_2 = \{y, v \,|\, \varphi = 2\pi - \varphi_0, (y, v) \in R^2\}$ on the side $\varphi < 2\pi - \varphi_0$ at the point $M_1(y_1, v_1) \in G_v^1$; the surface W^s intersects the plane P_1 and P_2 along, respectively, the curves N_1^s ($y = y_1^s(v)$) and N_2^s ($y = y_2^s(v)$) going off to infinite and dividing each of the planes into two parts.

P r o o f. We consider the Liapunov function V in the domain $G_0 = \{\varphi, y, v \,|\, \varphi_0 \leqslant \varphi \leqslant 2\pi - \varphi_0,\ (y, v) \in R^2\}$. Since according to (1.5), $F'(\varphi) \leqslant 0$ in G_0, the derivative $V^\cdot \leqslant 0$ and, consequently, system (2.1) does not have o-cycles in G_0 . Hence by virtue of that fact that coordinate φ increases (decreases) along the trajectories of system (2.1) when $y > 0$ ($y < 0$), while separatrix W_1^u (W_2^u) locally goes in domain G_v^1 (G_v^2), it follows that separatrix W_1^u (W_2^u) intersects the plane P_2 (P_1) at point M_1 (M_2). Let us consider the disposition of surface W^s in G_0 . By virtue of (1.5) a number $l_0 > 0$ exists (for example, $l_0 = 1$ for $F = \sin\varphi$ such that the estimate

$$l_2(\varphi) \leqslant F(\varphi) - \gamma \leqslant l_1(\varphi) \tag{3.1}$$
$$l_1(\varphi) \equiv -1 - \gamma + l_0(-\varphi + 2\pi - \varphi_0),\quad l_2(\varphi) \equiv 1 - \gamma + l_0(\varphi_0 - \varphi)$$

is valid for any $\varphi \in [\varphi_0, 2\pi - \varphi_0]$.

We consider the linear systems

$$\varphi^{\cdot} = y, \ y^{\cdot} = -l_i(\varphi) - (ab)^{-1/2}\delta y + a^{-1}bv \tag{3.2}$$
$$v^{\cdot} = -(ab)^{-1/2}(1-\delta)v - b^2 \ (\delta^2 - \delta + \lambda b)y + b^{-1}(ab)^{-1/2}(a - b - a\delta)l_i(\varphi), \ i = 1, 2$$

In parameter domain D each of the systems in (3.2) has a saddle equilibrium state each, whose separatrix surface is the plane W_i^s. Plane W_i^s joins P_1 and P_2 and divides each of them into two parts. By virtue of (3.1), when $y \neq 0$ the trajectories of system (2.1) intersect the plane W_1^s (W_2^s) on the side of decrease (increase) of coordinate y when $y > 0$ and on the side of increase (decrease) of coordinate y when $y < 0$. Since on the circle S^1 the equilibrium state O_2 lies between the equilibrium states of systems (3.2), the surface W^s is located in G_0 between W_2^s and W_1^s and, consequently, joins the planes P_1 and P_2, separating each of them into two parts.

From Lemmas 2.1 and 3.1 it follows that the separatrix W_1^u (W_2^u), intersecting plane P_2 (P_1) on the side $\varphi < 2\pi - \varphi_0$ ($\varphi > \varphi_0$), hits either onto plane P_1 (P_2) on the side $\varphi < \varphi_0$ ($\varphi > 2\pi - \varphi_0$) or, intersecting cylinder P_0, onto plane P_2 (P_1) on the side $\varphi > 2\pi - \varphi_0$ ($\varphi < \varphi_0$) or onto cylinder P_0 and can intersect P_0 and P_2 (P_0, and P_1) several times.

By $N_1^u(y_1^u, v_1^u)$ ($N_2^u(y_2^u, v_2^u)$) we denote the limit point for the points at which separatrix W_1^u (W_2^u) intersects plane P_1 (P_2) and by $N_*^u(y_*^u, v_*^u)$ ($N_0^u(y_0^u, v_0^u)$) we denote the limit point for the points at which W_1^u (W_2^u) intersects plane P_2 (P_1) on the side $\varphi > 2\pi - \varphi_0$ ($\varphi < \varphi_0$). We introduce the characteristic functions $\rho_j(q)$ and $\rho_*(q)$, $q = (a, b, \lambda, \gamma)$, whose signs uniquely determine the relative disposition of the separatrix manifolds by the following formulas

$$\rho_j(q) = \begin{cases} y_0^u - y_1^s(v_0^u), & j = 0 \\ y_j^u - y_j^s(v_j^u), & j = 1, 2 \end{cases}, \quad \text{if } N_j^u \text{ exists}$$
$$\rho_j(q) = (-1)^j \rho_j^\circ, \quad j = 0, 1, 2, \quad \text{if } N_j^u \text{ does not exist}$$

$$\rho_*(q) = \begin{cases} y_*^u - y_2^s(v^u_*), & \text{if } N_*^u \text{ exists} \\ \rho_*^\circ, & \text{if } N_*^u \text{ does not exist} \end{cases}$$

where $\rho_{1,2}^\circ, \rho_*^\circ > 0$ and $\rho_0^\circ < 0$, and ensure the continuity of $\rho_j(q)$ and $\rho_*(q)$ when N_j^u and N_*^u vanish. By virtue of the theorems on the continuous dependence of integral manifolds on the parameters the functions $\rho_j(q)$ and $\rho_*(q)$ are continuous. The following properties of the characteristic functions stem from Lemma 2.2:

$$\rho_1(q) < 0, \ \rho_0(q) < 0, \ q \in \Delta_1, \Delta_2 \tag{3.3}$$
$$\rho_2(q) > 0, \ \rho_*(q) > 0, \ q \in \Delta_i, \ i = 1, 2, 3, 4$$

We establish certain additional properties of characteristic functions $\rho_j(q)$ and $\rho_*(q)$ by using the matching system (2.7).

Lemma 3.2. The functions $\rho_j(q)$ and $\rho_*(q)$ satisfy the relations

$$\rho_2(q) > 0,\ q \in \Delta_5,\ \Delta_6;\ \rho_*(q) > 0,\ q \in \Delta_5 \qquad (3.4)$$
$$\rho_1(q) < 0,\ \rho_0(q) < 0,\ q \in \Delta_7$$

Proof. The surfaces W_i^α and W_i^ω formed by the separatrices of the matching system intersect the planes P_1 and P_2 (see Sect. 2). We introduce the notation

$$W_i^\alpha \cap P_{3-i} = N_i^\alpha(y = y_i^\alpha(v)),\quad W_i^\omega \cap P_i = N_i^\omega(y = y_i^\omega(v)),\quad i = 1, 2$$

Since the saddle O_2 on the circle S^1 lies to the left (to the right) of the saddle of system (2.7) when $q \in \Delta_5, \Delta_6\ (\Delta_7)$ while the trajectories of system (2.1) intersect the surfaces W_i^ω and W_i^α on the side of increase (decrease) of coordinate y, the inequalities

$$y_i > y_i^\alpha(v_i),\quad y_2^\omega(v) > y_2^s(v),\quad q \in \Delta_5, \Delta_6 \qquad (3.5)$$
$$y_2 < y_2^\alpha(v_2),\quad y_1^\omega(v) < y_1^s(v),\quad q \in \Delta_7$$

are valid. We establish (3.4) by using (3.5) and allowing for the relative disposition of surfaces W_i^α and W_i^ω the orientation of the vector field of system (2.1) on them and the property of decrease (increase) of coordinate along the trajectories when $y < 0$ $(y > 0)$.

The proof of Theorem 1 follows from (3.5) and (3.4), the properties of functions $\rho_j(q)$ and $\rho_*(q)$ when $\lambda = 0$ (see Theorem 1 in [10])and the Cauchy theorem on the zeros of continuous functions. When $b > a$ there obtains a merging of the three bifurcation surfaces Π_b^j at $\gamma = 0$ (see point p in Fig. 4(3)) by virtue of the symmetry of system (2.1) relative to the replacements $\gamma = -\gamma^\circ$, $\varphi = -\varphi^\circ$, $y = -y^\circ$, $v = -v^\circ$. To the merging of surfaces Π_b^j corresponds a contour Γ in phase space, composed of the O_2-, φ^1- and φ^2-loops.

Theorem 2. 1) When $b \geqslant a$ $(b > a,\ \gamma = 0)$ at least one φ^1 (φ^2)-cycle is generated as parameter λ grows from zero to infinity. 2) System (2.1) has at least one φ^1-cycle in the parameter domain Δ_5.

Proof. 1) By virtue of the disposition of manifolds W_1^u and W^s of system (2.1) when $\lambda = 0$ (see Theorem 1 in [10]) there exists a surface $w_1 = \{\varphi, y, v \mid \varphi \in S^1,\ y = y(\varphi, v) > 0\ (< 0),\ v \in G_v^1\ (G_v^2)\}$, periodic in φ, which the trajectories of system (2.1) intersect without contact on the side of increase (decrease) of coordinate y. When $\lambda = \varepsilon \ll 1$, by virtue of the continuous dependence of the solution of integral manifolds on the parameter, the vector field of system (2.1) on surface w_1 is oriented in the same way as when $\lambda = 0$. In G_v^1 (G_v^2) we construct a domain K_v^ω, periodic in φ and homeomorphic to a torus, whose boundary is composed of w_1 and of the following surfaces:

$$u_i = \{\varphi, y, v \mid \varphi \in S^1, \ y \in G_v, \ v = v_i^\circ\}, \ i = 1, 2$$
$$w_2 = \{\varphi, y, v \mid \varphi \in S^1, \ y = y_2^\circ (y_1^\circ), \ v \in G_v\}$$

We denote the surface $K_v^\omega \cap P \ (P = \{y, v \mid \varphi = \varphi^\circ = \text{const}, \ (y, v) \in G_v\})$ by g^s. By virtue of system (2.1) the inequality $\varphi_0^{\cdot} = y_0 > 0 \ (< 0)$, is valid for any trajectory $(\varphi_0(t), y_0(t), v_0(t)) \in K_v^\omega$ and the vector field on ∂K_v^ω is directed toward the interior of K_v^ω (Lemma 2.1). Therefore, the trajectories of system (2.1), which generates mapping T, take g^s into itself and consequently, system (2.1) has at least one φ^1 (φ^2)-cycle in domain K_v^ω According to [10], when $\lambda = 0$ system (2.1) does not have φ^1 (φ^2)-cycles when $b \geqslant a \ (b > a, \ \gamma = 0)$ while the surface w_2 goes off to infinity. Consequently, the φ^1 (φ^2)-cycle existing for $\lambda = \varepsilon$, goes off to infinity as $\varepsilon \to 0$.

2) By constructing a domain K_v^ω with boundary $\partial K_v^\omega = \partial G_v^1 \cup W^l (W^l$ is the surface formed by a cycle of the matching system (2.7)), we establish, completely analogously to the preceding, the existence of at least one φ^1-cycle of system (2.1) with parameter values from domain Δ_3.

N o t e. The existence of a φ^1 (φ^2)-cycle has been established here with the aid of a mapping of a disk into itself, which, in general, can have a complex nature, for instance, does not contain stable points. Allowing for this possibility, a set located in K_v^ω is called a stable set φ_s^1 (φ_s^2) (see Definition 2 of a "stable φ-cycle" in [11]). The cycles existing in accordance with Theorem 1 belong to set φ_s^1 or φ_s^2 or coincide with it if there is only one of them.

The stable set φ_s^1 vanishes as the parameters vary from the domains $\lambda = \varepsilon \ll 1$ and Δ_5 to the domains $\Delta_1, \Delta_3, \Delta_7$. Consequently, a bifurcation set, corresponding to the vanishing of φ_s^1, exists. Since by Lemma 2.1, φ_s^1 cannot vanish at infinity, the vanishing of set φ_s^1 is connected, for the points of parameter domain σ^s, with a bifurcation of the φ^1-loop, of domain σ^u, with a bifurcation of cycles, and of domain σ^c, with a bifurcation of the vanishing of the complex structure in a neighborhood of the φ^1-loop. By similarly examining the behavior of set φ_s^2, existing for $\lambda = \varepsilon$ (Theorems 2 and 3 from [10]) and vanishing for parameter values from domains $\Delta_i \ (i = 1, 2, \ldots, 6)$, as well as the behavior of the o-cycle, existing by virtue of (2.6) for $\varepsilon > m(\gamma)(b - a) - \lambda > 0$ and vanishing for $\lambda = 0$, we establish the additional bifurcations, different from Π_b^j, described in Sect. 1. Under the assumption that the C_b^j are single-valued, the additional bifurcations in domain σ^u are bifurcation surfaces of multiple cycles (Fig. 4 in Sect. 1).

REFERENCES

1. S h a k h g i l ' d i a n, V. V. and L i a k h o v k i n, A. A., Systems for the Phase Self-tuning of Frequency. Moscow, "Sviaz'", 1972.

2. B a u t i n, N. N., Qualitative study of a certain equation of the theory of phase automatic frequency control. PMM Vol. 34, No. 5, 1970.

3. Beliustina, L. N. and Belykh, V. N., Qualitative investigation of a dynamic system on a cylinder. Differents. Uravnen., Vol. 9, No. 3, 1973.

4. Shil'nikov, L. P., Bifurcation theory of dynamic systems with homoclinic Poincaré curves. VII Internat. Konf. Nichtlineare Schwingungen, Bd. 1–2. Berlin, Akad. –Verlag, 1977.

5. Bakaev, Iu. N., Synchronizing properties of a third-order phase system of automatic frequency control. Radiotekhnika i Electronika, Vol. 10, No. 6, 1965.

6. Leonov, G. A., The second Liapunov method in the theory of phase synchronization. PMM Vol. 40, No. 2, 1976.

7. Leonov, G. A., Stability and oscillations of phase systems. Sib. Mat. Zh., Vol. 16, No. 5, 1975.

8. Safonov, V. M., Phase self-tuning of frequency with a second-order filter. Nauchn. Dokl. Vyssh. Shkoly, Radiotekhnika i Elektronika, No. 4, 1958.

9. Beliustina, L. N. and Bykov, V. V., On bifurcations and certain qualitative characteristics of a system of phase automatic frequency control with a second-order filter. Proc. Sympos. Appl. Math. and Cybernetics. Moscow, "Nauka", 1973.

10. Belykh, V. N. and Nekorkin, V. I., Qualitative investigation of a system of three differential equations from the theory of phase synchronization. PMM Vol. 39, No. 4, 1975.

11. Belykh, V. N. and Nekorkin, V. I., On the qualitative investigation of a multidimensional phase system. Sib. Mat. Zh., Vol. 18, No. 4, 1977.

12. Shil'nikov, L. P., On the generation of a periodic motion from a trajectory doubly-asymptotic to a saddle-type equilibrium state. Mat. Sb., Vol. 77, No. 3, 1968.

13. Shil'nikov, L. P., On one case of existence of a denumerable set of periodic motions. Dokl. Akad. Nauk SSSR, Vol. 160, No. 3, 1965.

14. Bautin, N. N., Behavior of Dynamic Systems Close to the Boundary of the Stability Domain. Moscow –Leningrad, Gostekhizdat, 1949.

Translated by N. H. C.